DUALITY IN PROJECTIVE GEOMETRY.

By N. J. Lennes.

Veblen and Young have given a set of independent assumptions for projective geometry.* Their assumptions are stated in terms of the abstract (undefined) symbols *point* and classes of points called *lines*. The plane, three-space and spaces of higher dimensions are defined as classes of points determined by certain collinearities. The definitions of three-space and of spaces of higher dimensions are direct generalizations of the definition of plane.† Since in the treatment of Veblen and Young the plane is defined as a class of points while point is an undefined symbol it is clear that the development cannot involve point and plane in precisely the same manner from the start. Indeed a considerable body of theorems must be proved before the general theorem of duality can be established. By a direct generalization of the theorems just mentioned and their proofs duality in spaces of higher dimensions is established.

The treatment of Veblen and Young has the obvious advantage that a small number of undefined symbols is used and that consequently the number of assumptions is small. On the other hand, it would seem desirable to treat point and plane as space duals in the assumptions themselves. While this requires a somewhat larger body of axioms it avoids some rather intricate and slippery argumentations at the very outset. This consideration will be of greater importance in case the assumptions are used as basis for a first course in projective geometry.

The purpose of this paper is to give an independent set of assumption which shall be sufficient for what Veblen and Young call general projective space.‡

The undefined elements are *point* and *plane,* each equally abstract and fundamental. Thus plane is not regarded as a class of points and the line does not occur explicitly at all in the assumptions. Point and plane are connected by two undefined relations "point *on* plane" and "plane *on* point." These are entirely independent except as noted on page 15. Thus "Point A is on plane α" need not mean "Plane α is on point A."

*A set of Assumptions for Projective Geometry, American Journal of Mathematics, vol. XXX (1908), pp. 347–380.

†See Veblen and Young, Projective Geometry, vol. I, pp. 29–33.

‡American Journal, vol. XXX, p. 347.

In general points are indicated by Roman Caps as A, and planes by small Greek letters as α.

1. A Set of Fundamental Propositions.

I_1. *The class of points contains at least three elements.*

I_2. *The class of planes contains at least three elements.*

II_1. *Three points are on at least one plane.*

II_2. *Three planes are on at least one point.*

III_1. *On three planes there is at least one point.*

III_2. *On three points there is at least one plane.*

IV_1. *If points A and B are on a plane α and C not on α, then A, B, C are on not more than one plane.*

IV_2. *If planes α and β are on a point A and γ not on A, then α, β, γ are on not more than one point.*

V_1. *Two points are on at least three planes.*

V_2. *Two planes are on at least three points.*

Definitions. Points which are on two planes are *collinear*.

Planes which are on two points are *collinear*.

VI_1. *Not all points on any one plane are collinear.*

VI_2. *Not all planes on any one point are collinear.*

Propositions $\mathrm{I}_1 \cdots \mathrm{VI}_2$ are clearly dual with respect to point and plane. That is, the propositions remain unchanged if the words *point* and *plane* are interchanged. Hence any proposition which is a formal consequence of these propositions remains valid when the words point and plane are interchanged.*

For proofs of the independence of this set of propositions see § 3.

2. Theorems.

The theorems of this section are all formal consequences of the assumptions of §1.

(1) Theorem: *Not all points are on any one plane.*

Proof. Suppose all points on a plane α. By I_1 there are at least two points A and B on α and by V_1 A and B are on a plane β distinct from α. By VI_1 there is a point C on β and not on α. Hence not all points are an α.

(2) Theorem: *If points A, B, C are on each of the distinct planes α and β and if A and B are on a third plane γ, then C is on γ.*

*For a discussion of formal deduction from assumptions stated in terms of abstract symbols see E. V. Huntington, The Fundamental laws of Addition and Multiplication in Elementary Algebra, Annals of Math., Second series, vol. 8 (1906), pp. 1–49. Especially pp. 2–4.

Proof. If C is not on γ then the points A, B, C are on only one plane (IV_1) contrary to the hypothesis that A, B, C are on each of the distinct planes α and β.

(3) THEOREM: *If the points A, B, C, D are on the plane α and if no three of them are collinear then there is one and only one point E on α such that A, B, E and also C, D, E are collinear.*

Proof: By (1) a point F exists which is not on α and by II_1, IV_1 the points A, B, F are on one and only oneplane β and C, D, F are on one and only one plane γ.

By III_1 there is at least one point E on the planes α, β, γ.

If there is another point E' on α, β, γ then by IV_1 E, E', F are on only one plane. That is β and γ are the same plane and A, B, C, D are collinear.

(4) THEOREM: *If the points A, B, E are collinear and also C, D, E are collinear then A, B, C, D are on the same plane.*

Proof: Let A, B, C be on the plane α (II_1) and let F be a point not on α (1). Let B, E, F be on a plane β and D, E, F on a plane γ. Since A, B, E are collinear they must be on both α and β (2). Similarly C, D, E are on both α and γ. Hence A, B, C, D are all on α.

(5) THEOREM: (*a*) *Any two points are collinear.* (*b*) *If points A, B, C and also A, B, D are collinear then A, B, C, D are collinear.*

Proof: (*a*) is a direct consequence of V_1. (*b*) Let A, B, C be on two planes α and β, and let A, B, D be on planes α' and β'. Then by (2) C is on α' and also on β'. Hence A, B, C, D are collinear.

It will be noted that the above theorems are all stated in terms of "point on plane," and that in the proofs only assumptions with subscripts 1 are used. The duals of these theorems are:

1′. *Not all planes are on any one point.*

2′. *If planes α, β, γ are on each of the distinct points A and B and if α and β are on a third point C then γ is on C.*

3′. *If the planes α, β, γ, δ are on the point A and if no three of them are collinear then there is one and only one plane ϵ on A such that α, β, ϵ and also γ, δ, ϵ are collinear.*

4′. *If the planes α, β, ϵ are collinear and also γ, δ, ϵ are collinear then α, β, γ, δ are on the same point.*

5′. (*a*) *Any two planes are collinear.* (*b*) *If planes α, β, γ and also α, β, δ are collinear then α, β, γ, δ are collinear.*

3. Consistency of the Assumptions.

It is customary to inquire whether a given set of assumptions is consistent, independent and categorical.*

*For discussions of the properties of a set of assumptions denoted by these terms see Huntington, loc. cit.; Veblen, Trans. Am. Math. Society, vol. 5 (1904), pp. 343–384, especially p. 346, and references cited in these papers.

That a set of assumptions is consistent is proved by exhibiting a concrete system in which the assumptions are satisfied and which for some reason is regarded as self-consistent. The assumptions of this paper are satisfied by a set of fifteen planes and fifteen points with seven points on every plane and seven planes on every point. In this system "a plane is on a point" if "the point is on the plane" and conversely. In the following each column of seven letters represents a plane and each letter represents a point.

(A)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
A	A	A	B	B	C	C	D	D	E	E	F	F	G	G
B	B	B	C	C	D	D	E	E	F	F	G	G	A	A
C	H	M	K	N	I	H	O	M	L	K	J	H	I	K
D	D	D	E	E	F	F	G	G	A	A	B	B	C	C
E	I	L	L	H	K	J	H	K	I	O	O	L	J	H
F	J	N	M	I	L	M	J	I	N	H	K	I	O	L
G	K	O	J	O	O	N	L	N	J	M	N	M	M	N

To the set of assumptions given above further assumptions may be added in a variety of ways to give more special geometries. Thus the last four assumptions of the Veblen-Young set may be added to make it categorical as an ordinary projective geometry.

4. Independence of the Assumptions.

To prove that the propositions $I_1 \cdots VI_2$ of § 1 form an independent set of assumptions it is only necessary to prove I_1, II_1, III_1, IV_1, V_1, VI_1 independent, for it follows by the same arguments that I_2, II_2, $\cdots$, VI_2 are independent among themselves. This establishes the complete independence of the whole system; for the propositions with subscripts 1 cannot be used to prove a proposition with subscript 2 or vice versa since one set is stated in terms of "point on plane" and the other in terms of "plane on point." The propositions to be proved independent are:

I_1. The class of points contains at least three elements.

II_1. Three points are on at least one plane.

III_1. On three planes there is at least one point.

IV_1. If points P_1 and P_2 are on a plane α and P_3 not on α then P_1, P_2, P_3 are on not more than one plane.

V_1. Two points are on at least three planes.

VI_1. Not all points on any one plane are collinear.

These are shown to be independent by the following systems.

I_1. A system in which no point exists.

Propositions II_1, $\cdots$, VI_1 are to be regarded as hypothetical as regards the existence of points and planes. Thus II_1 written out in full would read "If points and planes exist three points are on at least one plane."

II_1. A system consisting of all planes in a projective geometry which are on one or more of seven points no four of which are coplanar and of all points on these planes.

III_1. A system consisting of the planes and points of Euclidian Geometry.

IV_1. A system consisting of system (A) given in §2, with the point K "on" the plane (3). Then ABK are on the distinct planes (2) and (3) but K is not on the plane (1) while A and B are on this plane.

V_1. A system in which the vertices of a tetrahedron $ABCD$ are points and any triad of these points are planes.

VI_1. A system consisting of three planes on each of which are the points A, B, C.

The assumptions of §1 together with the theorems deduced from them in §2 are sufficient to characterize a general projective space with the one exception that it does not follow from §1 that if a set of points is "on" a given plane then that plane is "on" those points. That is, "P_1 on α_1" need not mean "α_1 on P_1."

If now we define "P_1 on α_1" to mean the same as "α_1 on P_1" then the assumptions of §1 are far from independent. That they are still consistent follows from the finite system (A), p. 14. The redundancy of the assumptions in the presence of a reciprocal "on" was to be expected *a priori*, in as much as in that case the properties given in $I_1 \cdots VI_1$ in terms of "point on plane" certainly determine some properties in terms of "plane on point."

With a reciprocal "on" propositions I_1, I_2, II_1, II_2, IV_1, V_1, VI_1 form an independent set of assumptions while the remaining propositions are consequences of these. This we now proceed to prove.

III_1 and III_2 are immediate consequences of II_2 and II_1 respectively.

Proof of IV_2. Given a point A on planes α and β but not on γ. Suppose two points B and C on each of the planes α, β, γ. Since B and C are on γ but A not on γ, then A, B, C are on only one plane (IV_1), contrary to the assumption that A, B, C are on α and β.

Proof of V_2. Given any two planes α and β. By I_2, II_2 α and β are on a point A. Let D be a point on α but not on β (VI_1) and E a point on β but not on α. By V_1, three planes γ, δ, ϵ are on D and E. On α, β, γ there is a point A' (II_2); on α, β, δ a point B'; and on α, β, ϵ a point C'. Hence on α, β are the three points A', B', C'. If A', B', C' were not distinct at least two of the three planes γ, δ, ϵ would fail to be distinct.

Proof of VI_2. We need to show that on a given point A on which are two given planes α and β there is a plane γ not collinear with α and β. Let B be a point in α and not in β and C a point in β but not in α. Then the plane γ on A, B, C is not collinear with α and β.

To establish the independence of the remaining propositions of this set

we note that the systems given above to prove the independence of I_1, II_1, IV_1, V_1, VI_1 are sufficient to prove the independence of these propositions in the new set.

We next prove the independence of I_2, II_2.

I_2. A system containing no planes.

II_2. A system consisting of the points and planes of ordinary projective geometry with one point removed.

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